Galois theory for analogical classifiers

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Analogical proportion: a is to b as c is to d

Two key cognitive processes: Inference and Creativity

Applications in ML & AI

- NLP & translation [L03, S05,M20,A21a,M22]
- **Classification & recom.** [B07,H16,B17,C17 C18,C20b,C22b]
- **Case-based & Machine reasoning** [F89,G83,L19a,L19b,L21,M21]
- Transfer learning [B19,C20a, A21b]
- VisualQA, ScholasticAP, Explainability [S15,P19,H20]
- ...

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Outline

1. Brief overview: Formal models of analogy
2. Analogy based classification: Analogical inference principle (AIP)
3. Analogy-preserving (AP) functions
4. Galois theory of analogical classifiers
5. Application: Boolean analogical classifiers

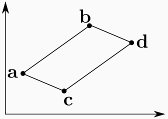
Analogical proportions: Classical examples

- Geometric proportion ($\mathbb{R}$):

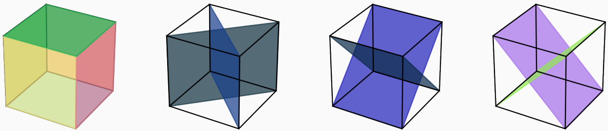
$$a \times d = b \times c$$

- Arithmetic proportion ($\mathbb{R}^m$):

$$\mathbf{a} - \mathbf{b} = \mathbf{c} - \mathbf{d}$$



- Boolean proportion ($\mathbb{B}^m$): particular case



NB: simultaneously capture similarities and dissimilarities

Formalizing analogies: axiomatic approach

Definition: An *analogy* over a nonempty set X is a 4-ary relation R over X that satisfies 3 axioms [L03,M08]

1. **Reflexivity:** $R(a, b, a, b)$.
2. **Symmetry:** $R(a, b, c, d) \implies R(c, d, a, b)$.
3. **Central permutation:** $R(a, b, c, d) \implies R(a, c, b, d)$.

Immediate consequences: $\forall a, b, c, d \in X$

1. $R(a, a, b, b)$ (*identity*)
2. $R(a, b, c, d) \implies R(d, b, c, a)$ (*extreme permutation*)
3. $R(a, b, c, d) \implies R(b, a, d, c)$ (*inside pair reversing*)
4. $R(a, b, c, d) \implies R(d, c, b, a)$ (*complete reversal*)

Comment: Could be argued...conceptual spaces...

Examples of analogy models

Minimal model: $R_{\mathcal{M}} = \{(x,x,y,y),(x,y,x,y)\}$

NB: In the Boolean case $X = \mathbb{B} = \{0,1\}$

$$R_{\mathcal{M}} := \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Klein model [K82]: $R_{\mathcal{K}}(a,b,c,d)$ if $b - a = d - c$

NB: In the Boolean case $X = \mathbb{B} = \{0,1\}$

$$R_{\mathcal{K}} := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

NB: It fulfills the additional property $R_{\mathcal{K}}(a,b,\neg a,\neg b)$

Formalizing analogies: further approaches

Relational: $R(a, b, c, d) \equiv P(P_1(a, b), P_1(c, d))$, for P, P_1 predicates

Example: $R(\mathit{wine}, \mathit{France}, \mathit{beer}, \mathit{Germany})$

Functional: $R(a, b, c, d)$ if $b = T(a)$ and $d = T(c)$, for some T

Example: $R(\mathit{go}, \mathit{went}, \mathit{make}, \mathit{made})$

Model Theoretic: Two main (and somewhat similar) approaches: based on *Structure mapping theory* **[G83,F89]** and *Justifications* **[A20]**

Example: Equation $R(20, 4, 30, x)$ has a clear solution $x = 6$.
NB: $x = 9$ is another since $R(20, 4, 30, 9) = R(10 \cdot 2, 2^2, 10 \cdot 3, 3^2)$.

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Formalizing analogies: settings

NB: Different ways to define analogies depending on the underlying structure of X and **the task** at hand...

Examples: arbitrary sets, lattices, matrices, words and sentences, preferences, graphs, images, ...

Here: Classification over $\mathbf{X} = X^n$

Given: a set $\mathbf{X}$ of instances (objects) and a set Y of labels (classes)

Find: a classifier $f : \mathbf{X} \to Y$

Extension: $R(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ holds $\Leftrightarrow \forall i \in [1, m]$, $R(a_i, b_i, c_i, d_i)$ holds.

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Analogical inference principle (AIP)

Let R and S be analogy models on $\mathbf{X}$ and Y, resp.

Analogical Inference Principle (AIP):

$$\frac{R(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})}{S(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), f(\mathbf{d}))}$$

NB: If the label of $\mathbf{d}$ is unknown, then we can infer it by solving the equation $S(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), x)$

Main Problem: Given analogies R and S be analogy models on $\mathbf{X}$ and Y, resp., for which classifiers is the AIP sound?

As we will see: they are the analogy-preserving functions

Dually: Given a class of classifiers, for which analogy models R and S is the AIP sound?

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Definition of AP functions

A function $f : \mathbf{X} \to Y$ is *analogy-preserving* relative to (R, S) if for all $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbf{X} = X^n$, the following implication holds:

$$\big(R(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \ \& \ S\text{-}\mathrm{solv}(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}))\big) \implies S(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), f(\mathbf{d})),$$

where $S\text{-}\mathrm{solv}(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}))$ if $\exists x \in Y$ with $S(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), x)$.

In other words: AP functions are exactly those for which AIP never fails. They are referred to as **analogical classifiers**

Problem: Describe the class of AP functions!

Notation: Denote by $\mathrm{AP}(R, S)$ the set of all AP relative to (R, S).

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Key observation

Observation: Every polarity $\rhd A \times B$ induces a Galois connection between A and B: for $C \subseteq A$ and $D \subseteq B$,

- $\varphi_{AB}(C) = \{b \in B \colon a \rhd b, \text{ for every } a \in C\}$, and
- $\varphi_{BA}(D) = \{a \in A \colon a \rhd b, \text{ for every } b \in D\}$

Thus: analogy preservation induces a Galois framework between classifiers and pairs of analogies!

In particular: they constitute closure systems!

Question: What are the Galois closed sets of classifiers and corresponding Galois closed sets of analogy pairs?

NB: This is exactly the "Main Problem" and its dual!

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Preliminaries: functions, operations and clones

Notation and terminology:

- $\mathcal{F}_{XY}^{(n)} := \{f \mid f\colon X^n \to Y\}$ and $\mathcal{F}_{XY} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_X^{(n)}$.
- For $C \subseteq \mathcal{F}_{XY}$, let $C^{(n)} := C \cap \mathcal{F}_{XY}^{(n)}$.
- $\mathcal{O}_X^{(n)} := \mathcal{F}_{XX}^{(n)}$ and $\mathcal{O}_X := \bigcup_{n\in\mathbb{N}} \mathcal{O}_X^{(n)}$.
- $\mathcal{J}_X$ is the set of all projections on X, *i.e.*, maps $(a_1,\dots,a_n) \mapsto a_i$.
- Composition of a set $C \subseteq \mathcal{F}_{YZ}$ with a set $K \subseteq \mathcal{F}_{XY}$ is

$$CK := \{f(g_1,\dots,g_n) \in \mathcal{F}_{XZ} \mid f \in C^{(n)}, g_1,\dots,g_n \in K^{(m)}\},$$

where $f(g_1,\dots,g_n) \in \mathcal{F}_{XZ}^{(m)}$ is defined by the rule

$$f(g_1,\dots,g_n)(\mathbf{a}) := f(g_1(\mathbf{a}),\dots,g_n(\mathbf{a})) \quad \text{for all } \mathbf{a} \in X^m.$$

- $C \subseteq \mathcal{O}_X$ is a clone if $\mathcal{J}_X \subseteq C$ and $CC \subseteq C$. **(E.Post!)**

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Known Galois theories: operations and relations

Notation and terminology:

- $\mathcal{R}_X^{(m)}$ denotes the set of all $R \subseteq X^m$, and $\mathcal{R}_X := \bigcup_{m\in\mathbb{N}} \mathcal{R}_X^{(m)}$.
- $f \in \mathcal{O}_X^{(n)}$ preserves $R \in \mathcal{R}_X^{(m)}$ (or R is invariant under f), $f \triangleright R$, if for all $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R$, we have $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in R$, where

$$f(\mathbf{a}_1, \ldots, \mathbf{a}_n) := (f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1m}, \ldots, a_{nm})).$$

- $\triangleright$ induces a **Galois connection** Pol – Inv between $\mathcal{O}_X$ and $\mathcal{R}_X$:

$$\operatorname{Pol}\mathcal{R} := \{f \in \mathcal{O}_A \mid \forall R \in \mathcal{R}\colon f \triangleright R\},$$
$$\operatorname{Inv}\mathcal{F} := \{R \in \mathcal{R}_A \mid \forall f \in \mathcal{F}\colon f \triangleright R\}.$$

Theorem [S78, P79]:

A set $\mathcal{C} \subseteq \mathcal{O}_X$ is definable by a set $\mathcal{R} \subseteq \mathcal{R}_X$ (*i.e.*, $C = \operatorname{Pol}\mathcal{R}$) **iff** C is a locally closed clone. (**Explain!**)

NB: The closed sets of relations, are known as relational clones...

Known Galois theories: functions and constraints

Notation and terminology:

- $\mathcal{R}_{XY}^{(m)}$ denotes the set of all (R, S) (constraints), where $R \subseteq X^m$ and $S \subseteq X^m$, and $\mathcal{R}_{XY} := \bigcup_{m \in \mathbb{N}} \mathcal{R}_{XY}^{(m)}$.
- $f \in \mathcal{F}_{XY}^{(n)}$ preserves $(R, S) \in \mathcal{R}_{XY}^{(m)}$, $f \triangleright (R, S)$, if for all $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R$, we have $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in S$.

  Example: The set of decreasing functions is defined by $(\leq_X, \geq_Y)$.

  NB: if $f \triangleright (R, S)$, then $f \triangleright (R', S')$ where $R' \subseteq R$ and $S \subseteq S'$.
- $\triangleright$ induces a **Galois connection** Pol – Inv between $\mathcal{F}_{XY}$ and $\mathcal{R}_{XY}$:

$$\operatorname{Pol} \mathcal{Q} := \{f \in \mathcal{F}_{XY} \mid \forall (R, S) \in \mathcal{Q} \colon f \triangleright (R, S)\},$$

$$\operatorname{Inv} \mathcal{F} := \{(R, S) \in \mathcal{R}_{XY} \mid \forall f \in \mathcal{F} \colon f \triangleright (R, S)\}.$$

NB: $\operatorname{Pol}\{R\} = \operatorname{Pol}\{(R, R)\}$

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NB: $\mathrm{Pol}\{R\} = \mathrm{Pol}\{(R, R)\}$

Known Galois theories: functions and constraints (cont.)

Theorem [C05]:

A set $\mathcal{C} \subseteq \mathcal{F}_{XY}$ is definable by a set $\mathcal{Q} \subseteq \mathcal{R}_{XY}$ **iff**
C is a locally closed minion, *i.e.*, $C\mathcal{J}_X \subseteq C$. (**Explain!**)
Moreover: they are exactly the sets definable by functional equations.

NB: they are downsets w.r.t. the minor quasi-order: $f \preceq g$ **if** $f \in g\mathcal{J}_X$

NB2: The dual Galois sets were also described in [C05]

Known Galois theories: functions and constraints (cont.)

Let: $\mathcal{R}_{XY}^{(C_1,C_2)}$ be the set of all (C_1, C_2)-constraints, *i.e.*, (R, S) where R and S invariant under clones C_1 and C_2, respectively.

Theorem [C09]:

A set $\mathcal{C} \subseteq \mathcal{F}_{XY}$ is definable by a set $\mathcal{Q} \subseteq \mathcal{R}_{XY}^{(C_1,C_2)}$ **iff** C is a locally closed (C_1, C_2)-clonoid, *i.e.*, $CC_1 \subseteq C$ **and** $C_2C \subseteq C$.

NB1: (C_1, C_2)-clonoids are downsets for $f \preceq_{(C_1,C_2)} g$ **if** $f \in C_2\{g\}C_1$

NB2: When C_1 and C_2 are the clones of affine functions, (C_1, C_2)-clonoids are exactly the sets definable by linear functional eq.s

NB3: The dual Galois sets were also described in [C09]

Back to analogical classifiers (AP functions)

Recall: For $(R, S) \in \mathcal{A}_X \times \mathcal{A}_Y$**...**

$f\colon \mathbf{X} \to Y$ AP relative to (R, S) if for all $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbf{X} = X^n$:

$$\big(R(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \;\&\; S\text{-}\mathrm{solv}(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}))\big) \implies S(f(\mathbf{a}), f(\mathbf{b}), f(\mathbf{c}), f(\mathbf{d})),$$

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Proposition [C22b]

For analogies R and S, we have $\mathrm{AP}(R, S) = \mathrm{Pol}(R, S')$, where

$$S' := S \cup \{(a, b, c, d) \in Y^4 \mid \nexists x \in Y\colon (a, b, c, x) \in S\}.$$

In particular: $\mathrm{AP}(R, S) = \mathrm{Pol}(R, S')$ is a locally closed minion.

Notation: $\mathcal{A}'_Y := \{S' \mid S \in \mathcal{A}_Y\}$ and $\mathcal{A}_{XY} := \mathcal{A}_X \times \mathcal{A}'_Y$, and $\mathcal{A}_{XY}^{(C_1, C_2)} := \mathcal{A}_{XY} \cap \mathcal{R}_{XY}^{(C_1, C_2)}$. **NB:** Analogies contain constant tuples...

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Example

Recall the two basic Boolean models of analogy:

$$R_{\mathcal{M}} := \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad R_{\mathcal{K}} := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Thus: $R'_{\mathcal{M}} = R_{\mathcal{M}} \cup \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and $R'_{\mathcal{K}} = R_{\mathcal{K}}$.

NB: $(R_{\mathcal{M}}, R'_{\mathcal{M}})$ is a relaxation of $(R_{\mathcal{K}}, R'_{\mathcal{K}}) = (R_{\mathcal{K}}, R_{\mathcal{K}})$

Describing Galois closed sets of analogical classifiers

Notation and terminology:

- Let $\mathcal{R} \subseteq \mathcal{R}_X$. An $\mathcal{R}$-locality is any matrix $D \sqsubseteq R \in \mathcal{R}$.
- For $\mathcal{Q} \subseteq \mathcal{R}_{XY}$, let $\mathcal{Q}_1 := \{R \in \mathcal{R}_X \mid \exists S \in \mathcal{R}_Y \textbf{ s.t. } (R, S) \in \mathcal{Q}\}$
- A set $C \subseteq \mathcal{F}_{XY}$ is $\mathcal{Q}$-locally closed if for all $f \in \mathcal{F}_{XY}$ we have $f \in C$ whenever for every $\mathcal{Q}_1$-locality D, either
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$\mathcal{C} \subseteq \mathcal{F}_{XY}$ is definable by analogical (C_1, C_2)-constraints **iff** it is a (C_1, C_2)-analogically locally closed (C_1, C_2)-clonoid.

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Application: analogical classifiers w.r.t. $R_{\mathcal{M}}$ and $R_{\mathcal{K}}$

Corollary [C22b]

Let L denote the clone of affine functions. We have
$\mathrm{AP}(R_{\mathcal{K}}, R_{\mathcal{K}}) = \mathrm{AP}(R_{\mathcal{K}}, R_{\mathcal{M}}) = \mathrm{AP}(R_{\mathcal{M}}, R_{\mathcal{K}}) = \mathrm{AP}(R_{\mathcal{M}}, R_{\mathcal{M}}) = \mathsf{L}$.

NB1: Since $R_{\mathcal{K}} = R'_{\mathcal{K}}$, $\mathrm{AP}(R_{\mathcal{K}}, R_{\mathcal{K}}) = \mathrm{Pol}(R_{\mathcal{K}}, R'_{\mathcal{K}}) = \mathrm{Pol}\, R_{\mathcal{K}} = \mathsf{L}$.

NB2: Since $(R_{\mathcal{M}}, R'_{\mathcal{M}})$ is a relaxation of $(R_{\mathcal{K}}, R_{\mathcal{K}})$,

$$\mathsf{L} = \mathrm{Pol}(R_{\mathcal{K}}, R_{\mathcal{K}}) \subseteq \mathrm{Pol}(R_{\mathcal{M}}, R'_{\mathcal{M}}) = \mathrm{AP}(R_{\mathcal{M}}, R_{\mathcal{M}})$$

NB3: Both $R_{\mathcal{M}}$ and $R_{\mathcal{K}}$ are invariant under $\mathcal{I}$ (constant preserving).

Thus: If $f \in \{g\}\mathcal{I}$ and $g \notin \mathrm{AP}(R, S)$, then $f \notin \mathrm{AP}(R, S)$
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Proof of $\mathsf{AP}(R_{\mathcal{M}}, R_{\mathcal{M}}) = \mathsf{L}$

NB4: We only need to consider binary functions!

NB5: up to permutation of arguments, the binary Boolean funct.s are:

- the constant 0 and 1 functions,
- the first projection $\mathrm{pr}_1 : (x_1, x_2) \mapsto x_1$ and its negation $\neg_1 = \overline{\mathrm{pr}_1}$,
- the conjunction $\wedge$ and its negation $\uparrow$,
- the disjunction $\vee$ and its negation $\downarrow$,
- the implication $\rightarrow$ and its negation $\nrightarrow$, and
- the addition $+$ modulo 2 and its negation $\leftrightarrow$.

NB6: Every $g \notin \mathsf{L}$ has an $\mathcal{I}$-minor in $\{\wedge, \vee, \uparrow, \downarrow, \nrightarrow, \rightarrow\}$.

Proof of $\mathsf{AP}(R_{\mathcal{M}}, R_{\mathcal{M}}) = \mathsf{L}$

To complete: it suffices to show $\mathsf{AP}(R_{\mathcal{M}}, R_{\mathcal{M}}) \cap \{\wedge, \vee, \uparrow, \downarrow, \nrightarrow, \rightarrow\} = \varnothing$

- $\wedge, \vee \notin \mathsf{Pol}(R_{\mathcal{M}}, R'_{\mathcal{M}})$ because

$$\wedge \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \notin R'_{\mathcal{M}}, \qquad \vee \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \notin R'_{\mathcal{M}}.$$

- $\uparrow, \downarrow \notin \mathsf{Pol}(R_2, R'_2)$ because

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The cases $\mathrm{AP}(R_{\mathcal{K}}, R_{\mathcal{M}}) = \mathrm{AP}(R_{\mathcal{M}}, R_{\mathcal{K}}) = \mathrm{L}$ are shown similarly.

Final remarks and perspectives

[C17]: First obtained $\mathrm{AP}(R_{\mathcal{M}}, R_{\mathcal{M}}) = \mathrm{L}$ by an ad-hoc approach

[C18]: We showed $P(\mathrm{err}_f > 0) \leq 4 \cdot d(f, \mathrm{L})$ (unif. dist. over $2^{\{0,1\}^m}$)

[C20]: Extension to nominal domains (but still w.r.t. $R_{\mathcal{M}}$)

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Explore: other conceptual domains, analogy models and error bounds

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That's all folks...

Merci de votre attention!

Obrigado pela vossa atenção!

Thank you for your attention!

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Proof: Galois closed sets of analogical classifiers

Theorem [C22b]

$\mathcal{C} \subseteq \mathcal{F}_{XY}$ is definable by analogical (C_1, C_2)-constraints **iff** it is a (C_1, C_2)-analogically locally closed (C_1, C_2)-clonoid.

Proof (Necessity): By [C09], $\mathcal{C}$ is a (C_1, C_2)-clonoid.

To see that $\mathcal{C}$ is (C_1, C_2)-analogically locally closed, let $f \notin \mathcal{C}$ (n-ary).

Hence: there is $(R, S) \in \mathcal{A}_{XY}^{(C_1,C_2)}$ **s.t.** $\forall g \in \mathcal{C}$, $g \triangleright (R, S)$, but $f \not\triangleright (R, S)$.

Consider the locality $D = (\mathbf{a}_1, \dots, \mathbf{a}_n) \sqsubseteq R$ **s.t.** $f(\mathbf{a}_1, \dots, \mathbf{a}_n) \notin S$.

Since every $g \in \mathcal{C}^{(n)}$ satisfies (R, S), we have that $fD \neq gD$.

Also: $D \sqsubseteq R$ and $S \in \{S_0 \in \mathcal{R}_B \mid (R, S_0) \in \mathcal{A}_{XY}^{(C_1,C_2)}, \mathcal{C}R \subseteq S_0\}$, and we have $fR \not\subseteq S$, since $fD \notin S$.

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NB: The set of such analogical (C_1, C_2)-constraints defines $\mathcal{C}$!

As $\mathcal{C}$ is (C_1, C_2)-analog. locally closed, $\exists$ $(\mathcal{A}_{XY}^{(C_1,C_2)})_1$-locality D **s.t.**

- $fD \neq gD$, $\forall g \in \mathcal{C}^{(n)}$, **and**
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